

Towards constructing Tsallis' q-Coherent States

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Abstract

In this manuscript we define q-coherent states corresponding to the Tsallis' statistics. For them and for the ordinary coherent states we calculate the respective distributions of probabilities. For the latter, we encounter some results that are not too well known and deserve better dissemination.

Keywords: Coherent States, q-Coherent States.

1 Introduction

During more than 25 years, an important topic in statistical mechanics theory revolved around the notion of generalized q -statistics, pioneered by Tsallis [1]. It has been amply demonstrated that, in many occasions, the celebrated Boltzmann-Gibbs logarithmic entropy does not yield a correct description of the system under scrutiny [2]. Other entropic forms, called q -entropies, produce a much better performance [2]. One may cite a large number of such instances. For example, non-ergodic systems exhibiting a complex dynamics [2].

The non-extensive statistical mechanics of Tsallis' has been employed to fruitfully discuss phenomena in variegated fields. One may mention, for instance, high-energy physics [3]-[4], spin-glasses [5], cold atoms in optical lattices [6], trapped ions [7], anomalous diffusion [8], [9], dusty plasmas [10], low-dimensional dissipative and conservative maps in dynamical systems [11], [12], [13], turbulent flows [14], Levy flights [15], the QCD-based Nambu, Jona, Lasinio model of a many-body field theory [16], etc. Notions related to q -statistical mechanics have been found useful not only in physics but also in chemistry, biology, mathematics, economics, and informatics [17], [18], [19].

Here we wish to contribute to a line of enquiry that has been the focus of continuous research activity for several years: the exploration of nonlinear, q -versions of some of the fundamental equations of physics. A recent example is the new nonlinear Schrödinger equation advanced by Nobre, Rego Monteiro and Tsallis [20]. In this work, with regards to the line of enquiry just mentioned, we revisit the important subject of q -coherent states $|\alpha, q\rangle$, initiating the effort of constructing them. As a preliminary step, in reviewing some details of the ordinary coherent states, we encounter some not too well known results that seem to deserve wider publicity.

2 Coherent States

2.1 Preliminaries

Our protagonists here will be the coherent states of the harmonic oscillator (HO) $|\alpha\rangle$, or Glauber states [21, 22, 23]. A coherent state $|\alpha\rangle$ is a specific kind of quantum state, the one that most resembles a classical state. It is applicable to the quantum harmonic oscillator, the electromagnetic field, etc.,

and describes a maximal kind of coherence and a classical kind of behavior. The states $|\alpha\rangle$ are normalized, i.e., $\langle\alpha|\alpha\rangle = 1$, and they provide us with a resolution of the identity operator

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1, \quad (2.1)$$

which is a completeness relation for the coherent states [23]. The standard coherent states $|\alpha\rangle$ for the harmonic oscillator are eigenstates of the annihilation operator \hat{a} , with complex eigenvalues

$$\alpha = \frac{q + ip}{\sqrt{2}}, \quad (2.2)$$

which satisfy $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [23].

The n -th HO eigenfunction is

$$\phi_n(x) = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} \mathcal{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad (2.3)$$

where \mathcal{H}_n is Hermite's n -th order generalized function

$$\mathcal{H}_n(x) = \left(\pi^{\frac{1}{2}} 2^n n!\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x), \quad (2.4)$$

while H_n is the concomitant Hermite polynomial. In the x -representation, the coherent state reads

$$\psi_\alpha(x) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \phi_n(x), \quad (2.5)$$

or

$$\psi_\alpha(x) = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mathcal{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right). \quad (2.6)$$

For convenience we choose $\sqrt{\frac{m\omega}{\hbar}} = 1$. Thus, for the HO we have

$$\phi_n(x) = \mathcal{H}_n(x), \quad (2.7)$$

and for its coherent states (CS)

$$\psi_\alpha(x) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mathcal{H}_n(x). \quad (2.8)$$

2.2 Some not too well known results

We tackle now some issues that are not usually expounded in text books, starting by proving that the CS read [See Appendix A for a deduction of this formula (Eq. (A.12))]

$$\psi_\alpha(x) = \pi^{-\frac{1}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}. \quad (2.9)$$

Now we expand (2.9) à la Hermite

$$\psi_\alpha(x) = \sum_{n=0}^{\infty} a_n \mathcal{H}_n(x), \quad (2.10)$$

and compute a_n as

$$a_n = \int_{-\infty}^{\infty} \psi_\alpha(x) \mathcal{H}_n(x) dx. \quad (2.11)$$

Accordingly,

$$a_n = \pi^{-\frac{1}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x} \mathcal{H}_n(x) dx, \quad (2.12)$$

that can be recast as

$$a_n = \frac{\pi^{-\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}}}{\left(n! 2^n \pi^{\frac{1}{2}}\right)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{\alpha}{\sqrt{2}}\right)^2} \mathcal{H}_n(x) dx. \quad (2.13)$$

We appeal now to an Integral-Table result (see [25]) to obtain

$$a_n = \frac{\pi^{-\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}}}{\left(n! 2^n \pi^{\frac{1}{2}}\right)^{\frac{1}{2}}} \pi^{\frac{1}{2}} 2^{\frac{n}{2}} \alpha^n, \quad (2.14)$$

or

$$a_n = (n!)^{-\frac{1}{2}} \alpha^n e^{-\frac{|\alpha|^2}{2}}. \quad (2.15)$$

Replacing now (2.15) into (2.10) we reach (2.8) and prove (2.9). We could not find such equation in the literature available to us, so that there is a possibility that (2.9) might be a new result.

We evaluate now the Wigner function for a coherent state, starting from (Cf. (2.2))

$$W_{\psi}(p, q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi\left(q - \frac{x}{2}\right) \psi^*\left(q + \frac{x}{2}\right) e^{ipx} dx, \quad (2.16)$$

that for CS is

$$\psi_{\alpha}\left(q - \frac{x}{2}\right) = \pi^{-\frac{1}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{q^2}{2}} e^{\sqrt{2}\alpha q} e^{-\frac{x^2}{8}} e^{\frac{qx}{2}} e^{-\frac{\alpha x}{\sqrt{2}}}, \quad (2.17)$$

$$\psi_{\alpha}^*\left(q + \frac{x}{2}\right) = \pi^{-\frac{1}{4}} e^{-\frac{\alpha^{*2}}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{q^2}{2}} e^{\sqrt{2}\alpha^* q} e^{-\frac{x^2}{8}} e^{-\frac{qx}{2}} e^{\frac{\alpha^* x}{\sqrt{2}}}. \quad (2.18)$$

From these states we find for Wigner's function

$$W_{\psi_{\alpha}}(p, q) = \frac{1}{2\pi^{\frac{3}{2}}} e^{-\frac{\alpha^2}{2}} e^{-\frac{\alpha^{*2}}{2}} e^{-|\alpha|^2} e^{-q^2} e^{\sqrt{2}q(\alpha+\alpha^*)} \otimes \int_{-\infty}^{\infty} e^{-\frac{x^2}{8}} e^{\frac{-(\alpha-\alpha^*)x}{\sqrt{2}}} e^{ipx} dx. \quad (2.19)$$

Appealing now to the Integral-Table result [26] we find

$$W_{\psi_{\alpha}}(p, q) = \frac{1}{\pi} e^{-(p^2+q^2)} e^{\sqrt{2}(q-ip)\alpha} e^{\sqrt{2}(q+ip)\alpha^*} e^{-2|\alpha|^2}. \quad (2.20)$$

Noting that $\sqrt{2} \beta = q + ip$, (2.20) becomes

$$W_{\psi_{\alpha}}(\beta) = \frac{1}{\pi} e^{-2|\alpha-\beta|^2}, \quad (2.21)$$

a result that differs by a factor two from that of [27]. This happens because the measure used by Wigner was $dp \wedge dq$ instead of $d^2\alpha$, in common use nowadays.

As shown above

$$I = \frac{1}{\pi} \iint_{-\infty}^{\infty} |\alpha > d^2\alpha < \alpha| \quad (2.22)$$

with

$$|\alpha > = \pi^{-\frac{1}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x} |x > dx. \quad (2.23)$$

Thus, any square integrable function verifies

$$|\varphi\rangle = \frac{1}{\pi} \iint_{-\infty}^{\infty} |\alpha\rangle d^2\alpha a(\alpha), \quad (2.24)$$

where $a(\alpha) = \langle \alpha | \varphi \rangle$. Thus,

$$\varphi(x) = \frac{1}{\pi} \iint_{-\infty}^{\infty} a(\alpha) \psi_\alpha(x) d^2\alpha. \quad (2.25)$$

Here $\varphi(x) = \langle x | \varphi \rangle$, whose Wigner form reads

$$W_\varphi(p, q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi\left(q - \frac{x}{2}\right) \varphi^*\left(q + \frac{x}{2}\right) e^{ipx} dx. \quad (2.26)$$

Replacing (2.25) into (2.26) we get

$$W_\varphi(p, q) = \frac{1}{2\pi^3} \iiint_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\alpha_1) a^*(\alpha_2) \psi_{\alpha_1}\left(q - \frac{x}{2}\right) \psi_{\alpha_2}^*\left(q + \frac{x}{2}\right) \otimes e^{ipx} d\alpha_1 d\alpha_2 dx. \quad (2.27)$$

Integrating over x we encounter

$$W_\varphi(p, q) = \frac{1}{\pi^3} \iiint_{-\infty}^{\infty} e^{-\frac{\alpha_1^2}{2}} e^{-\frac{|\alpha_1|^2}{2}} e^{-\frac{\alpha_2^{*2}}{2}} e^{-\frac{|\alpha_2|^2}{2}} e^{-q^2} e^{\sqrt{2}(\alpha_1 + \alpha_2^*)q} e^{\left(\frac{\alpha_1 - \alpha_2^*}{\sqrt{2}} - ip\right)^2} \otimes a(\alpha_1) a^*(\alpha_2) d\alpha_1 d\alpha_2, \quad (2.28)$$

the general expression for Wigner's function for an arbitrary (square-integrable) function, written in terms of coherent states.

We compute now the probability distribution (PD) associated to a CS, starting with *the overlap between a plane wave of momentum k and $|\alpha\rangle$*

$$\langle k | \alpha \rangle = 2^{-\frac{1}{2}} \pi^{-\frac{3}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x} e^{-ikx} dx. \quad (2.29)$$

Using again the Integral-Table result [26] we obtain

$$\langle k|\alpha \rangle = \pi^{-\frac{1}{4}} e^{\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{k^2}{2}} e^{-i\sqrt{2}\alpha k}. \quad (2.30)$$

Accordingly, the associated PD is

$$|\langle k|\alpha \rangle|^2 = \pi^{-\frac{1}{2}} e^{\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{k^2}{2}} e^{-i\sqrt{2}\alpha k} e^{\frac{\alpha^{*2}}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{k^2}{2}} e^{i\sqrt{2}\alpha^* k}, \quad (2.31)$$

that can be simplified to

$$|\langle k|\alpha \rangle|^2 = \pi^{-\frac{1}{2}} e^{-\frac{(\alpha^* - \alpha + i\sqrt{2}k)^2}{2}}. \quad (2.32)$$

Noting that $\sqrt{2}\alpha = q + ip$, (2.32) becomes

$$|\langle k|\alpha \rangle|^2 = \pi^{-\frac{1}{2}} e^{-(k-p)^2}. \quad (2.33)$$

This the PD associated to a plane wave of momentum k and the CS $|\alpha \rangle$ is just a Gaussian.

3 Towards q -Coherent States

We start here work in this respect, and wish to report some advances. The q -coherent state is obtained by replacing the exponential (2.9) by the associated q -exponential $e_q(x)$ [2]

$$e_q(x) = [1 + (1 - q)x]^{1/1-q}; \quad q \in \mathcal{R}, \quad (3.1)$$

that becomes the ordinary exponential at $q = 1$. Accordingly, we have

$$\psi_{\alpha q}(x) = A(q, \alpha) \left[1 + \frac{q-1}{2} (x^2 - 2\sqrt{2}\alpha x + |\alpha|^2 + \alpha^2) \right]^{\frac{1}{1-q}}, \quad (3.2)$$

where $A(q, \alpha)$ is a constant to be determined. We need, first of all, an explicit expression for the overlap involved in the normalization process

$$\langle \psi_{\alpha q} | \psi_{\alpha q} \rangle = A^2(q, \alpha) \int_{-\infty}^{\infty} \left[1 + \frac{q-1}{2} (x^2 - 2\sqrt{2}\alpha x + |\alpha|^2 + \alpha^2) \right]^{\frac{1}{1-q}} \otimes$$

$$\left[1 + \frac{q-1}{2} \left(x^2 - 2\sqrt{2}\alpha^*x + |\alpha|^2 + \alpha^{*2}\right)\right]^{\frac{1}{1-q}} dx. \quad (3.3)$$

We recast (3.3) in the form

$$\begin{aligned} \langle \psi_{\alpha q} | \psi_{\alpha q} \rangle = & A^2(q, \alpha) \left(\frac{q-1}{2}\right)^{\frac{2}{1-q}} \int_{-\infty}^{\infty} \left(x - \sqrt{2}\alpha - \sqrt{|\alpha|^2 + \alpha^2 - \frac{2}{q-1}}\right)^{\frac{1}{1-q}} \otimes \\ & \left(x - \sqrt{2}\alpha + \sqrt{|\alpha|^2 + \alpha^2 - \frac{2}{q-1}}\right)^{\frac{1}{1-q}} \otimes \\ & \left(x - \sqrt{2}\alpha^* - \sqrt{|\alpha|^2 + \alpha^{*2} - \frac{2}{q-1}}\right)^{\frac{1}{1-q}} \otimes \\ & \left(x - \sqrt{2}\alpha^* + \sqrt{|\alpha|^2 + \alpha^{*2} - \frac{2}{q-1}}\right)^{\frac{1}{1-q}} dx. \end{aligned} \quad (3.4)$$

Utilizing Eq. (B.3) from the Appendix B we find

$$\begin{aligned} \langle \psi_{\alpha q} | \psi_{\alpha q} \rangle = & A^2(q, \alpha) \frac{q-1}{5-q} \left(\frac{q-1}{2}\right)^{\frac{2}{1-q}} \otimes \\ & F_D \left(\frac{5-q}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{4}{q-1}; \right. \\ & 1 + \sqrt{2}\alpha + \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}}, \\ & 1 + \sqrt{2}\alpha - \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}}, \\ & 1 + \sqrt{2}\alpha^* + \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}}, \\ & \left. 1 + \sqrt{2}\alpha^* - \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}} \right). \end{aligned} \quad (3.5)$$

Now, because of the normalization requirement

$$\langle \psi_{\alpha q} | \psi_{\alpha q} \rangle = 1, \quad (3.6)$$

we get for the constant $A(q, \alpha)$ the expression

$$\begin{aligned}
A(q, \alpha) &= \left[\frac{q-1}{5-q} \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \right. \\
&\quad {}_F_D \left(\frac{5-q}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{4}{q-1}; \right. \\
&\quad \left. 1 + \sqrt{2}\alpha + \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}}, \right. \\
&\quad \left. 1 + \sqrt{2}\alpha - \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}}, \right. \\
&\quad \left. 1 + \sqrt{2}\alpha^* + \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}}, \right. \\
&\quad \left. 1 + \sqrt{2}\alpha^* - \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}} \right]^{-\frac{1}{2}}. \tag{3.7}
\end{aligned}$$

In similar fashion we compute the scalar product (overlap) of two arbitrary CS

$$\begin{aligned}
\langle \psi_{\alpha q} | \psi_{\beta q} \rangle &= A(q, \alpha) A(q, \beta) \frac{q-1}{5-q} \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \otimes \\
&\quad {}_F_D \left(\frac{5-q}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{4}{q-1}; \right. \\
&\quad \left. 1 + \sqrt{2}\alpha + \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}}, \right. \\
&\quad \left. 1 + \sqrt{2}\alpha - \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}}, \right. \\
&\quad \left. 1 + \sqrt{2}\beta^* + \sqrt{\beta^{*2} - |\beta|^2 - \frac{2}{q-1}}, \right. \\
&\quad \left. 1 + \sqrt{2}\beta^* - \sqrt{\beta^{*2} - |\beta|^2 - \frac{2}{q-1}} \right). \tag{3.8}
\end{aligned}$$

We pass now to the PD associated to a q -coherent state. We start by noting that

$$|\alpha, q\rangle = A(q, \alpha) \int_{-\infty}^{\infty} \left[1 + \frac{q-1}{2} (x^2 - 2\sqrt{2}\alpha x + |\alpha|^2 + \alpha^2) \right]^{\frac{1}{1-q}} |x\rangle dx. \quad (3.9)$$

Thus, the overlap between a plane wave of momentum k and $|\alpha, q\rangle$ is

$$\langle k|\alpha, q\rangle = \frac{A(q, \alpha)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[1 + \frac{q-1}{2} (x^2 - 2\sqrt{2}\alpha x + |\alpha|^2 + \alpha^2) \right]^{\frac{1}{1-q}} dx, \quad (3.10)$$

that can be rewritten as

$$\begin{aligned} \langle k|\alpha, q\rangle = & \frac{A(q, \alpha)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[x + \sqrt{2}\alpha + \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}} \right]^{\frac{1}{1-q}} \otimes \\ & \left[x + \sqrt{2}\alpha - \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}} \right]^{\frac{1}{1-q}} dx. \end{aligned} \quad (3.11)$$

Using now the Integral-Table result [28] we find

$$\begin{aligned} \langle k|\alpha, q\rangle = & \frac{\text{Sgn}(k)\sqrt{2\pi}A(q, \alpha)|k|^{\frac{3-q}{q-1}} e^{-\frac{i\pi\text{Sgn}(k)}{q-1}} \otimes \\ & e^{i\left(\sqrt{2}\alpha + \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}}\right)} \phi\left(\frac{1}{q-1}, \frac{2}{q-1}; -2i\sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}} |k|\right). \end{aligned} \quad (3.12)$$

The PD we are looking for becomes

$$\begin{aligned} |\langle k|\alpha, q\rangle|^2 = & \frac{2\pi[A(q, \alpha)]^2 |k|^{\frac{6-2q}{q-1}} \otimes}{\left[\Gamma\left(\frac{2}{q-1}\right)\right]^2} \otimes \\ & e^{i\left[\sqrt{2}(\alpha - \alpha^*) + \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}} - \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}}\right]} \otimes \end{aligned}$$

$$\begin{aligned} & \phi \left(\frac{1}{q-1}, \frac{2}{q-1}; -2i \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}} |k| \right) \otimes \\ & \phi \left(\frac{1}{q-1}, \frac{2}{q-1}; 2i \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}} |k| \right). \end{aligned} \quad (3.13)$$

We can calculate now $\langle x^2 \rangle_q$. It is given by

$$\begin{aligned} \langle x^2 \rangle_q = A^2(q, \alpha) \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \int_{-\infty}^{\infty} \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^{*2} + \frac{2}{q-1} \right)^{\frac{1}{1-q}} \otimes \\ x^2 \left(x^2 - 2\sqrt{\alpha}x + |\alpha|^2 + \alpha^2 + \frac{2}{q-1} \right)^{\frac{1}{1-q}} dx \end{aligned} \quad (3.14)$$

Lets $\beta_1, \beta_2, \beta_3, \beta_4$ be given by

$$\begin{aligned} \beta_1 &= \sqrt{2}\alpha^* + \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}} \\ \beta_2 &= \sqrt{2}\alpha^* - \sqrt{\alpha^{*2} - |\alpha|^2 - \frac{2}{q-1}} \\ \beta_3 &= \sqrt{2}\alpha + \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}} \\ \beta_4 &= \sqrt{2}\alpha - \sqrt{\alpha^2 - |\alpha|^2 - \frac{2}{q-1}} \end{aligned} \quad (3.15)$$

Then we can write (3.14) as

$$\begin{aligned} \langle x^2 \rangle_q = A^2(q, \alpha) \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \int_{-\infty}^{\infty} x^2 (x - \beta_1)^{\frac{1}{1-q}} (x - \beta_2)^{\frac{1}{1-q}} \otimes \\ (x - \beta_3)^{\frac{1}{1-q}} (x - \beta_4)^{\frac{1}{1-q}} dx \end{aligned} \quad (3.16)$$

Appealing now to (B.3) we obtain for (3.16)

$$\begin{aligned} \langle x^2 \rangle_q &= 2A^2(q, \alpha) \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \frac{\Gamma\left(\frac{7-3q}{q-1}\right)}{\Gamma\left(\frac{4}{q-1}\right)} \\ F_D \left(\frac{7-3q}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}; \frac{4}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) \end{aligned} \quad (3.17)$$

In the same way we have for $\langle x \rangle_q$ the expression

$$\begin{aligned} \langle x \rangle_q &= A^2(q, \alpha) \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \frac{\Gamma\left(\frac{6-2q}{q-1}\right)}{\Gamma\left(\frac{4}{q-1}\right)} \\ F_D \left(\frac{6-2q}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}; \frac{4}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) \end{aligned} \quad (3.18)$$

The evaluation of $\langle p^2 \rangle_q$ is somewhat more involved. For it we have

$$\begin{aligned} \langle p^2 \rangle_q &= -A^2(q, \alpha) \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \int_{-\infty}^{\infty} \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^{*2} + \frac{2}{q-1} \right)^{\frac{1}{1-q}} \otimes \\ &\quad \frac{\partial^2}{\partial x^2} \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^2 + \frac{2}{q-1} \right)^{\frac{1}{1-q}} dx \end{aligned} \quad (3.19)$$

or

$$\begin{aligned} \langle p^2 \rangle_q &= -\frac{A^2(q, \alpha)}{(1-q)} \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \left[2 \int_{-\infty}^{\infty} \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^{*2} + \frac{2}{q-1} \right)^{\frac{1}{1-q}} \otimes \right. \\ &\quad \left. \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^2 + \frac{2}{q-1} \right)^{\frac{q}{1-q}} dx + \right. \\ &\quad \left. \frac{q}{1-q} \int_{-\infty}^{\infty} \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^{*2} + \frac{2}{q-1} \right)^{\frac{1}{1-q}} (2x - 2\sqrt{2}\alpha)^2 \otimes \right. \end{aligned}$$

$$\left(x^2 - 2\sqrt{\alpha}x + |\alpha|^2 + \alpha^2 + \frac{2}{q-1} \right)^{\frac{2q-1}{1-q}} dx \quad (3.20)$$

Appealing again to (B.3) the result for $\langle p^2 \rangle_q$ is

$$\begin{aligned} \langle p^2 \rangle_q = & -\frac{A^2(q, \alpha)}{(1-q)} \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \left\{ 2 \frac{\Gamma\left(\frac{3+q}{q-1}\right)}{\Gamma\left(\frac{2q+2}{q-1}\right)} \otimes \right. \\ & F_D \left(\frac{3+q}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{q}{q-1}, \frac{q}{q-1}; \frac{2q+2}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) + \\ & \quad \frac{q}{1-q} \left[8 \frac{\Gamma\left(\frac{3+q}{q-1}\right)}{\Gamma\left(\frac{4q}{q-1}\right)} \otimes \right. \\ & F_D \left(\frac{3+q}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{2q-1}{q-1}, \frac{2q-1}{q-1}; \frac{4q}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) - \\ & \quad \left. 8\sqrt{2}\alpha \frac{\Gamma\left(\frac{2q+2}{q-1}\right)}{\Gamma\left(\frac{4q}{q-1}\right)} \otimes \right. \\ & F_D \left(\frac{2q+2}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{2q-1}{q-1}, \frac{2q-1}{q-1}; \frac{4q}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) + \\ & \quad \left. 8\alpha^2 \frac{\Gamma\left(\frac{3q+1}{q-1}\right)}{\Gamma\left(\frac{4q}{q-1}\right)} \otimes \right. \\ & \left. F_D \left(\frac{3q+1}{q-1}; \frac{1}{q-1}, \frac{1}{q-1}, \frac{2q-1}{q-1}, \frac{2q-1}{q-1}; \frac{4q}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) \right] \Bigg\} \quad (3.21) \end{aligned}$$

Analogously, we have for $\langle p \rangle_q$

$$\begin{aligned} \langle p \rangle_q = & -iA^2(q, \alpha) \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \int_{-\infty}^{\infty} \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^{*2} + \frac{2}{q-1} \right)^{\frac{1}{1-q}} \otimes \\ & \frac{\partial}{\partial x} \left(x^2 - 2\sqrt{\alpha}x + |\alpha|^2 + \alpha^2 + \frac{2}{q-1} \right)^{\frac{1}{1-q}} dx \quad (3.22) \end{aligned}$$

or

$$\begin{aligned} \langle p \rangle_q = & -\frac{iA^2(q, \alpha)}{1-q} \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \int_{-\infty}^{\infty} \left(x^2 - 2\sqrt{\alpha^*}x + |\alpha|^2 + \alpha^{*2} + \frac{2}{q-1} \right)^{\frac{1}{1-q}} \otimes \\ & (2x - 2\sqrt{2}\alpha) \left(x^2 - 2\sqrt{\alpha}x + |\alpha|^2 + \alpha^2 + \frac{2}{q-1} \right)^{\frac{q}{1-q}} dx \end{aligned} \quad (3.23)$$

Recourse to (B.3) again, we obtain

$$\begin{aligned} \langle p \rangle_q = & -\frac{iA^2(q, \alpha)}{1-q} \left(\frac{q-1}{2} \right)^{\frac{2}{1-q}} \left[2 \frac{\Gamma\left(\frac{4}{q-1}\right)}{\Gamma\left(\frac{2q+2}{q-1}\right)} \otimes \right. \\ & F_D \left(\frac{4}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{q}{q-1}, \frac{q}{q-1}, \frac{2q+2}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) - \\ & \left. 2\sqrt{2}\alpha \frac{\Gamma\left(\frac{3+q}{q-1}\right)}{\Gamma\left(\frac{2q+2}{q-1}\right)} \otimes \right. \\ & \left. F_D \left(\frac{3+q}{q-1}, \frac{1}{q-1}, \frac{1}{q-1}, \frac{q}{q-1}, \frac{q}{q-1}, \frac{2q+2}{q-1}; 1+\beta_1, 1+\beta_2, 1+\beta_3, 1+\beta_4 \right) \right] \end{aligned} \quad (3.24)$$

With the mean q -values thus obtained, we can calculate $(\Delta x)_q(\Delta p)_q$.

It is easy to see that there is a one-to-one mapping $|\alpha \rangle \Leftrightarrow |\alpha, q \rangle$ that immediately arises from the well known one-to-one mapping between q -exponentials and ordinary ones, entailing that one can write the unity operator as:

$$I = \int_{-\infty}^{\infty} |\alpha, q \rangle A(q, \alpha) \langle \alpha, q| d^2\alpha = \frac{1}{\pi} \int_{-\infty}^{\infty} |\alpha \rangle \langle \alpha| d^2\alpha \quad (3.25)$$

with $A(q, \alpha)$ an still unknown constant, where $\lim_{q \rightarrow 1} A(q, \alpha) = \frac{1}{\pi}$

Thus, for any q , the basis $\{|\alpha, q \rangle\}$ constitute an over complete basis.

4 Quantum uncertainty in the limit $q \rightarrow 1$

We will show now that $\lim_{q \rightarrow 1} (\Delta x)_q (\Delta p)_q = \Delta x \Delta p = \frac{1}{2}$. This is to the essence in order to ensure that our q -extension of coherent states makes sense. For this endeavor we use the approximation, for q close to one, of the q -exponential. It is easily seen that one has

$$[1 + i(q-1)z]^{\frac{1}{1-q}} = \left[1 - \frac{q-1}{2}z^2\right] e^{-iz}. \quad (4.1)$$

As a consequence of (4.1) we obtain

$$\begin{aligned} & \left[1 + \frac{q-1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)\right]^{\frac{1}{1-q}} = \\ & \left[1 + \frac{q-1}{8}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)^2\right] e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} = \\ & \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) e^{-\frac{\beta}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} \Big|_{\beta=1} \end{aligned} \quad (4.2)$$

The normalized q -coherent state reads, in this approximation,

$$\psi_{\alpha q}(x) = A^{-1}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) e^{-\frac{\beta}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} \Big|_{\beta=1} \quad (4.3)$$

Of course, $A(q, \alpha)$ needs evaluation. For this purpose we calculate

$$\begin{aligned} A^2(q, \alpha) &= \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ & \int_{-\infty}^{\infty} e^{-\frac{\beta}{2}(x^2 - 2\sqrt{2}\alpha^* x + \alpha^{*2} + |\alpha|^2)} e^{-\frac{\gamma}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \Big|_{\beta=\gamma=1} \end{aligned} \quad (4.4)$$

By recourse to the Integral-Table result given in [29] we then find

$$\begin{aligned} A^2(q, \alpha) &= \sqrt{\pi} \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ & \left[\frac{\sqrt{2}}{\sqrt{\beta + \gamma}} e^{-\frac{\beta}{2}(\alpha^{*2} + |\alpha|^2)} e^{-\frac{\gamma}{2}(\alpha^2 + |\alpha|^2)} e^{\frac{(\beta\alpha^* + \gamma\alpha)^2}{\beta + \gamma}} \right]_{\beta=\gamma=1}. \end{aligned} \quad (4.5)$$

We can thus write

$$A^2(q, \alpha) = \sqrt{\pi} + (q-1)f_1(\alpha) + (q-1)^2f_2(\alpha), \quad (4.6)$$

where f_1 and f_2 are non-singular functions of α . As a consequence,

$$A(q, \alpha) = \sqrt{\sqrt{\pi} + (q-1)f_1(\alpha) + (q-1)^2f_2(\alpha)}, \quad (4.7)$$

and

$$\lim_{q \rightarrow 1} A(q, \alpha) = \pi^{\frac{1}{4}}. \quad (4.8)$$

We now can write for $\langle x^2 \rangle_q$

$$\begin{aligned} \langle x^2 \rangle_q &= A^{-2}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ &\int_{-\infty}^{\infty} e^{-\frac{\beta}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} x^2 e^{-\frac{\gamma}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \big|_{\beta=\gamma=1} \end{aligned} \quad (4.9)$$

Using once more the Integral-Table [29], one has

$$\begin{aligned} \langle x^2 \rangle_q &= A^{-2}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ &\left\{ \frac{2^{\frac{3}{2}}}{(\beta + \gamma)^{\frac{3}{2}}} e^{-\frac{\beta}{2}(\alpha^{*2} + |\alpha|^2)} e^{-\frac{\gamma}{2}(\alpha^2 + |\alpha|^2)} e^{\frac{(\beta\alpha^* + \gamma\alpha)^2}{\beta + \gamma}} (2i)^{-2} \sqrt{\pi} H_2 \left[\frac{i(\beta\alpha^* + \gamma\alpha)}{\sqrt{\beta + \gamma}} \right] \right\}. \end{aligned} \quad (4.10)$$

As a consequence,

$$\langle x^2 \rangle_q = A^{-2}(q, \alpha) \left\{ \sqrt{\pi} \left[\frac{(\alpha + \alpha^*)^2}{2} + \frac{1}{2} \right] + (q-1)g_1(\alpha) + (q-1)^2g_2(\alpha) \right\}, \quad (4.11)$$

where g_1 and g_2 are non-singular functions of α . Thus,

$$\lim_{q \rightarrow 1} \langle x^2 \rangle_q = \frac{1}{2} + \frac{(\alpha + \alpha^*)^2}{2} = \langle x^2 \rangle. \quad (4.12)$$

Proceeding now in similar fashion for $\langle x \rangle_q$ we obtain

$$\langle x \rangle_q = A^{-2}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes$$

$$\int_{-\infty}^{\infty} e^{-\frac{\beta}{2}(x^2-2\sqrt{2}\alpha^*x+\alpha^{*2}+|\alpha|^2)} x e^{-\frac{\gamma}{2}(x^2-2\sqrt{2}\alpha x+\alpha^2+|\alpha|^2)} dx \big|_{\beta=\gamma=1} \quad (4.13)$$

According to the Integral-Table result [29],

$$\begin{aligned} < x >_q = A^{-2}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ & \left\{ \frac{2}{\beta + \gamma} e^{-\frac{\beta}{2}(\alpha^{*2}+|\alpha|^2)} e^{-\frac{\gamma}{2}(\alpha^2+|\alpha|^2)} e^{\frac{(\beta\alpha^*+\gamma\alpha)^2}{\beta+\gamma}} (2i)^{-1} \sqrt{\pi} H_1 \left[\frac{i(\beta\alpha^* + \gamma\alpha)}{\sqrt{\beta + \gamma}} \right] \right\}, \end{aligned} \quad (4.14)$$

or

$$< x >_q = A^{-2}(q, \alpha) \left\{ \sqrt{\pi} \left[\frac{\alpha + \alpha^*}{\sqrt{2}} \right] + (q-1)h_1(\alpha) + (q-1)^2 h_2(\alpha) \right\}, \quad (4.15)$$

where h_1 and h_2 are again non-singular functions of α . Accordingly,

$$\lim_{q \rightarrow 1} < x >_q = \frac{\alpha + \alpha^*}{\sqrt{2}} = < x >. \quad (4.16)$$

For $< p^2 >_q$ we have instead

$$\begin{aligned} < p^2 >_q = -A^{-2}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ & \int_{-\infty}^{\infty} e^{-\frac{\beta}{2}(x^2-2\sqrt{2}\alpha^*x+\alpha^{*2}+|\alpha|^2)} \frac{\partial^2}{\partial x^2} e^{-\frac{\gamma}{2}(x^2-2\sqrt{2}\alpha x+\alpha^2+|\alpha|^2)} dx \big|_{\beta=\gamma=1} \end{aligned} \quad (4.17)$$

or

$$\begin{aligned} < p^2 >_q = A^{-2}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ & \int_{-\infty}^{\infty} e^{-\frac{\beta}{2}(x^2-2\sqrt{2}\alpha^*x+\alpha^{*2}+|\alpha|^2)} [\gamma^2(x - \sqrt{2}\alpha)^2 - \gamma] e^{-\frac{\gamma}{2}(x^2-2\sqrt{2}\alpha x+\alpha^2+|\alpha|^2)} dx \big|_{\beta=\gamma=1} \end{aligned} \quad (4.18)$$

As in previous cases, according to Integral-Table result [29] we have

$$< p^2 >_q = A^{-2}(q, \alpha) \left\{ \sqrt{\pi} \left[\frac{1}{2} - \frac{(\alpha - \alpha^*)^2}{2} \right] + (q-1)k_1(\alpha) + (q-1)^2 k_2(\alpha) \right\}. \quad (4.19)$$

Here k_1 and k_2 are non-singular functions as well. Therefore,

$$\lim_{q \rightarrow 1} \langle p^2 \rangle_q = \frac{1}{2} - \frac{(\alpha - \alpha^*)^2}{2} = \langle p^2 \rangle. \quad (4.20)$$

In analogy with the above case we now also have

$$\begin{aligned} \langle p \rangle_q &= -iA^{-2}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \gamma^2}\right) \otimes \\ &\int_{-\infty}^{\infty} e^{-\frac{\beta}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} \frac{\partial}{\partial x} e^{-\frac{\gamma}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \Big|_{\beta=\gamma=1} \end{aligned} \quad (4.21)$$

and, after employing again the Integral-Table result [29],

$$\langle p \rangle_q = -iA^{-2}(q, \alpha) \left\{ \sqrt{\pi} \left[\frac{\alpha - \alpha^*}{\sqrt{2}} \right] + (q-1)l_1(\alpha) + (q-1)^2l_2(\alpha) \right\}, \quad (4.22)$$

where l_1 and l_2 are non-singular functions of α . Thus,

$$\lim_{q \rightarrow 1} \langle p \rangle_q = \frac{\alpha - \alpha^*}{i\sqrt{2}} = \langle p \rangle. \quad (4.23)$$

From (4.12), (4.16), (4.20), and (4.23), we obtain

$$\lim_{q \rightarrow 1} (\Delta x)_q (\Delta p)_q = \Delta x \Delta p = \frac{1}{2}. \quad (4.24)$$

For the q -distribution, with q close to 1, and using

$$|q, \alpha \rangle = A^{-1}(q, \alpha) \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \int_{-\infty}^{\infty} e^{-\frac{\beta}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} |x \rangle dx \Big|_{\beta=1} \quad (4.25)$$

we have

$$\begin{aligned} \langle k | q, \alpha \rangle &= \frac{A^{-1}(q, \alpha)}{\sqrt{2\pi}} \left(1 + \frac{q-1}{2} \frac{\partial^2}{\partial \beta^2}\right) \otimes \\ &\int_{-\infty}^{\infty} e^{-ikx} e^{-\frac{\beta}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \Big|_{\beta=1} \end{aligned} \quad (4.26)$$

Again, from the Integral-Table result [29], we can write

$$\langle k|q, \alpha \rangle = A^{-1}(q, \alpha) \left[e^{-\frac{1}{2}(k^2 + 2\sqrt{2}i\alpha k - \alpha^2 + |\alpha|^2)} + (q - 1)f(\alpha, k) \right], \quad (4.27)$$

where f is non-singular. Therefore,

$$\lim_{q \rightarrow 1} \langle k|q, \alpha \rangle = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(k^2 + 2\sqrt{2}i\alpha k - \alpha^2 + |\alpha|^2)} = \langle k|\alpha \rangle, \quad (4.28)$$

and, as a consequence,

$$\lim_{q \rightarrow 1} |\langle k|q, \alpha \rangle|^2 = |\langle k|\alpha \rangle|^2 = \pi^{-\frac{1}{2}} e^{-(k-p)^2}, \quad (4.29)$$

a nice result indeed!

5 Conclusions

We have introduced in this work q -coherent states and obtained some interesting preliminary results, although much work remains to be done.

- Firstly, we have revisited the ordinary coherent ones and found that

$$\psi_\alpha(x) = \pi^{-\frac{1}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}, \quad (5.1)$$

which might perhaps be a new result.

- We also obtained the Wigner function associated to a coherent state and determined in closed form the overlap between a plane wave of momentum k and the coherent state (5.1), from which one obtains by squaring an ensuing probability distribution. We have not encountered such result elsewhere.
- Afterwards, we determined the most important relationships governing q -coherent states.
- In particular, we find that, in the limit $q \rightarrow 1$, minimal uncertainty is attained, which constitutes a fundamental result.
- Note that the q -coherent states constitute an over complete basis for any q .

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Appendix A: Proof of Eq.(2.9)

It is very well known the annihilation operator for the one-dimensional harmonic oscillator is given by

$$\hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}} \quad (\text{A.1})$$

In the x -representation of Quantum Mechanics this operator is expressed via

$$\hat{a}(x) = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right) \quad (\text{A.2})$$

Thus, a coherent state is defined as the eigenfunction

$$\hat{a}(x)\psi_\alpha(x) = \frac{1}{\sqrt{2}} \left(x\psi_\alpha(x) + \frac{d\psi_\alpha(x)}{dx} \right) = \alpha\psi_\alpha(x), \quad (\text{A.3})$$

or, equivalently,

$$\frac{d\psi_\alpha(x)}{dx} = (\sqrt{2}\alpha - x)\psi_\alpha(x). \quad (\text{A.4})$$

The solution of (A.4) is

$$\psi_\alpha(x) = Ce^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}. \quad (\text{A.5})$$

The constant C can be evaluated using the normalization condition

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 \int_{-\infty}^{\infty} e^{-x^2} e^{\sqrt{2}(\alpha+\alpha^*)x} dx = 1. \quad (\text{A.6})$$

Accordingly,

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 e^{\frac{(\alpha+\alpha^*)^2}{2}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2} dx = 1. \quad (\text{A.7})$$

By recourse to the result given in the Table [29] we now obtain

$$\int_{-\infty}^{\infty} e^{-\left(x - \frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2} dx = \sqrt{\pi}. \quad (\text{A.8})$$

As a consequence,

$$C = \pi^{-\frac{1}{4}} e^{-\frac{(\alpha+\alpha^*)^2}{4}}. \quad (\text{A.9})$$

Thus, we have for $\psi_\alpha(x)$ the expression

$$\psi_\alpha(x) = \pi^{-\frac{1}{4}} e^{-\frac{(\alpha+\alpha^*)^2}{4}} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}, \quad (\text{A.10})$$

or, equivalently,

$$\psi_\alpha(x) = e^{i\alpha_R\alpha_I} \pi^{-\frac{1}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}, \quad (\text{A.11})$$

where $\alpha = \alpha_R + i\alpha_I$. As $e^{i\alpha_R\alpha_I}$ is an imaginary phase, it can be eliminated from (A.11) to finally obtain

$$\psi_\alpha(x) = \pi^{-\frac{1}{4}} e^{-\frac{\alpha^2}{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}. \quad (\text{A.12})$$

Appendix B: Lauricella functions

Here we need results from Reference [30]. Our subject is the fourth Lauricella function of four variables, given by [30]

$$\begin{aligned} F_D(a; b_1, b_2, b_3, b_4; c; x_1, x_2, x_3, x_4) = \\ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} \frac{(a)_{m_1+m_2+m_3+m_4} (b_1)_{m_1} (b_2)_{m_2} (b_3)_{m_3} (b_4)_{m_4}}{(c)_{m_1+m_2+m_3+m_4}} \otimes \\ \frac{x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}}{m_1! m_2! m_3! m_4!}. \end{aligned} \quad (\text{B.1})$$

This function satisfies [30]

$$\begin{aligned} \int_0^1 u^{a-1} (1-u)^{c-a-1} (1-ux_1)^{-b_1} (1-ux_2)^{-b_2} (1-ux_3)^{-b_3} (1-ux_4)^{-b_4} du \\ = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_D(a; b_1, b_2, b_3, b_4; c; x_1, x_2, x_3, x_4). \end{aligned} \quad (\text{B.2})$$

After two variables' changes we can deduce, from (B.2), the relation

$$\begin{aligned} & \int_0^1 u^{c-a-1} (1-u)^{b_1+b_2+b_3+b_4-c} (u_1+z_1)^{-b_1} (u_2+z_2)^{-b_2} (u_3+z_3)^{-b_3} (u_4+z_4)^{-b_4} du \\ &= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_D(a; b_1, b_2, b_3, b_4; c; 1-z_1, 1-z_2, 1-z_3, 1-z_4). \end{aligned} \quad (B.3)$$

Appendix C: Reviewing uncertainty relations for coherent states

For an ordinary coherent state $|\alpha\rangle$ we have:

$$\langle x^2 \rangle = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} x^2 e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (C.1)$$

With the use of the Integral-Table result [29] we find

$$\langle x^2 \rangle = (2i)^{-2} H_2 \left[\frac{i(\alpha^* + \alpha)}{\sqrt{2}} \right] \quad (C.2)$$

and then

$$\langle x^2 \rangle = \frac{1}{2} + \frac{(\alpha + \alpha^*)^2}{2} \quad (C.3)$$

For $\langle x \rangle$ the situation is quite similar

$$\langle x \rangle = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} x e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (C.4)$$

Using the Integral-Table result [29] again we obtain

$$\langle x \rangle = (2i)^{-1} H_1 \left[\frac{i(\alpha^* + \alpha)}{\sqrt{2}} \right] \quad (C.5)$$

and thus

$$\langle x \rangle = \frac{\alpha + \alpha^*}{\sqrt{2}} \quad (C.6)$$

For $\langle p \rangle$, the integral is somewhat more complicated

$$\langle p^2 \rangle = -\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} \frac{\partial^2}{\partial x^2} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (C.7)$$

or:

$$\langle p^2 \rangle = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} [1 - (x - \sqrt{2}\alpha)^2] e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (C.8)$$

Now, by recourse to the Integral-Table result [29] we obtain

$$\langle p^2 \rangle = 1 - 2\alpha^2 - i\sqrt{2}\alpha H_1 \left[\frac{i(\alpha^* + \alpha)}{\sqrt{2}} \right] + \frac{1}{4} H_2 \left[\frac{i(\alpha^* + \alpha)}{\sqrt{2}} \right] \quad (C.9)$$

or

$$\langle p^2 \rangle = \frac{1}{2} - \frac{(\alpha - \alpha^*)^2}{2} \quad (C.10)$$

For dealing with $\langle p \rangle$ one starts with

$$\langle p \rangle = -i\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} \frac{\partial}{\partial x} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (C.11)$$

or

$$\langle p \rangle = i\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha^*x + \alpha^{*2} + |\alpha|^2)} (x - \sqrt{2}\alpha) e^{-\frac{1}{2}(x^2 - 2\sqrt{2}\alpha x + \alpha^2 + |\alpha|^2)} dx \quad (C.12)$$

and, finally,

$$\langle p \rangle = \frac{\alpha - \alpha^*}{i\sqrt{2}} \quad (C.13)$$

Accordingly, the well-known uncertainty relation for a coherent state becomes

$$\Delta x \Delta p = \frac{1}{2}, \quad (C.14)$$

i.e., minimal uncertainty, the main feature of coherent states.